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Hawking Radiation for Non-minimally Coupled Matter from Generalized 2D Black Hole Models

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Abstract

It is well known that spherically symmetric reduction of General Relativity (SSG) leads to non-minimally coupled scalar matter. We generalize (and correct) recent results to Hawking radiation for a class of dilaton models which share with the Schwarzschild black hole non-minimal coupling of scalar fields and the basic global structure. An inherent ambiguity of such models (if they differ from SSG) is discussed. However, for SSG we obtain the rather disquieting result of a *negative* Hawking flux at infinity, if the usual recipe for such calculations is applied.

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1 Introduction

It has been known for a long time [1] that spherical reduction of d = 4 General Relativity (GR) for scalar matter leads to non-minimal coupling. We fix the radius to be a dimensionless quantity in the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{-2\phi} d\Omega^2 \quad . \tag{1}$$

The metric $g_{\mu\nu}$ refers to the variables $x^\mu = (x^0, x^1)$. With the dilaton field $\phi(x) = -\ln r$ the effective action in $d = 1 + 1$ for scalar matter f becomes

$$S_m = -\frac{1}{2} \int d^2x \sqrt{-g} e^{-2\phi} g^{\mu\nu} \partial_\mu f \partial_\nu f \quad . \tag{2}$$

Covariant derivatives will refer to the metric $g_{\mu\nu}$ in the following.

Somewhat surprisingly until very recent times [2–6] no computation of the Hawking radiation for that case seems to exist. The purpose of our present note is to give a comprehensive and direct answer to that question including all physically interesting models which generalize spherically symmetric gravity (SSG). This also allows us to improve the results of [2–6] and to show the arbitrariness involved when SSG is generalized. Therefore, in the following we take the geometric part of our action to be [7]

$$S_{gr} = \int d^2x \sqrt{-g} e^{-2\phi} (R + 4a(\nabla\phi)^2 + B e^{4(1-a)\phi}). \tag{3}$$

This class of dilaton models covers all asymptotically flat theories with one horizon and arbitrary power behavior in Kruskal coordinates at the singularity. It contains SSG as the special case $a = \frac{1}{2}$. The string-inspired dilaton black hole [8] follows for $a = 1$. For all $a \leq 1$ and $B > 0$ the global structure from (3) corresponds to the Penrose diagram of the Schwarzschild black hole, although a 'Schwarzschild gauge' is possible for the interval $0 < a \leq 1$ only with the limiting case of $a = 1$ [8] showing the potentially dangerous feature of a non-null incomplete but null complete singularity [7].

2 Conformal Anomaly

In the SSG case the (ultralocal) measure for the matter integration is well defined, because

$$\int d\Omega \sqrt{-{}^4 g} = e^{-2\phi}\sqrt{-g} \quad , \tag{4}$$

where ${}^4 g$ is the determinant of the 4 dimensional metric in GR. For the generalized class of models (3), however, this definition is not unique, as well as the one for an eventual non-minimal factor for the possible coupling to matter in (2). In that case we have to allow the general replacements $\phi \to \varphi(\phi)$ in (2) and $\phi \to \psi(\phi)$ in (4), where φ and ψ may be general (scalar) functions of the dilaton field. The measure $\mathcal{D}f$ is defined by requiring that $\int \mathcal{D}f \exp(i \int d^2x \sqrt{-g} e^{-2\psi(x)} f^2(x))$ is a field independent constant. With these replacements and in terms of the field $\tilde{f} = f\ e^{-\psi}$ which satisfies the standard normalization condition, (2) can be rewritten as

$$S = -\frac{1}{2}\int \sqrt{-g} d^2x\ \tilde{f} A \tilde{f} \quad , \tag{5}$$

where

$$A = -e^{-2\varphi+2\psi} g^{\mu\nu}(\nabla_\mu \nabla_\nu + 2(\psi_{,\mu} - \varphi_{,\mu})\nabla_\nu + \psi_{,\mu\nu} - 2\varphi_{,\mu}\psi_{,\nu}) \quad . \tag{6}$$

The path integral for $\tilde{f}$ leads to the effective action

$$W = \frac{1}{2} \mathrm{Tr} \ln A \quad . \tag{7}$$

After continuation to the Euclidean domain A becomes an elliptic second order differential operator. The corresponding one loop effective action W can be expressed in terms of the zeta function of the operator A [1]:

$$W = -\frac{1}{2}\zeta_A'(0), \qquad \zeta_A(s) = \mathrm{Tr}(A^{-s}) \tag{8}$$

Prime denotes differentiation with respect to s. From W regularized in this way an infinitesimal conformal transformation $\delta g_{\mu\nu} = \delta k g_{\mu\nu}$ produces the trace of the (effective) energy momentum tensor

$$\delta W = \frac{1}{2}\int d^2x \sqrt{g} \delta g^{\mu\nu} T_{\mu\nu} = -\frac{1}{2}\int d^2x \sqrt{g} \delta k(x) T^\mu_\mu(x) \tag{9}$$

[1]For a clear and extensive discussion on the ζ function technique consult the recent monograph [9].

Due to the multiplicative transformation property $\delta A = -\delta k A$ of (6) (valid in $d = 2$ only) with the definition of a generalized ζ-function

$$\zeta(s|\delta k, A) = \mathrm{Tr}(\delta k A^{-s}) \tag{10}$$

the variation in (9) can be identified with

$$\delta W = -\frac{1}{2}\zeta(0|\delta k, A) \quad , \tag{11}$$

where we used $\delta\zeta_{A_k}(s) = s\mathrm{Tr}(A^{-1}\delta k)$. Combining (11) and (9) we obtain

$$\zeta(0|\delta k, A) = \int d^2x\sqrt{g}\delta k(x)T^\mu_\mu(x)\ . \tag{12}$$

By a Mellin transformation one can show that $\zeta(0|\delta k, A) = a_1(\delta k, A)$ [10], where a_1 is defined as a coefficient in a small t asymptotic expansion of the heat kernel:

$$\mathrm{Tr}(F\exp(-At)) = \sum_n a_n(F, A)t^{n-1} \tag{13}$$

To evaluate a_1 we use the standard method [10]. To this end we represent A as

$$A = -(\hat{g}^{\mu\nu}D_\mu D_\nu + E), \qquad E = \hat{g}^{\mu\nu}(-\varphi_{,mu}\varphi_{,\nu} + \varphi_{,\mu\nu}) \tag{14}$$

where $\hat{g}^{\mu\nu} = e^{-2\varphi+2\psi}g^{\mu\nu}$, $D_\mu = \nabla_\mu + \omega_\mu$, $\omega_\mu = \psi_{,\mu} - \varphi_{,\mu}$. For a_1 follows [10]

$$a_1(\delta k, A) = \frac{1}{24\pi}\int d^2x\sqrt{-\hat{g}}\delta k(\hat{R} + 6E)\ . \tag{15}$$

Returning to the initial metric and comparing with (9) we obtain the most general form of the 'conformal anomaly' [2] for non-minimal coupling in d=2:

$$T^\mu_\mu = \frac{1}{24\pi}(R - 6(\nabla\varphi)^2 + 4\Box\varphi + 2\Box\psi) \tag{16}$$

[2]We use this expression although for a non-Weyl invariant geometric part (3) of the action is broken already at the classical level.

3 Hawking radiation

In a background given by (3) the matter action transforms simply under conformal transformations which may be used for important simplifications. In conformal gauge $g^{+-} = -2e^{-2\rho}$ (16) reduces to

$$T_{+-} = -\frac{1}{12\pi}(\partial_+\partial_-\rho + 3\partial_+\varphi\partial_-\varphi - 2\partial_+\partial_-\varphi - \partial_+\partial_-\psi)\ . \tag{17}$$

For SSG alone $\varphi^{SSG} = \psi^{SSG} = \phi$ are determined uniquely. In order to see the ambiguities involved for a 'generalized' theory it is sufficient to consider a general linear dependence on the dilaton field, $\varphi = \alpha\phi$, $\psi = \beta\phi$.

The Hawking flux is expressed in terms of T_{--} which can be obtained by integrating the conservation condition for the energy momentum tensor [11]

$$0 = \nabla^\mu T_{\mu-} = \partial_+ T_{--} + \partial_- T_{+-} - 2(\partial_-\rho)T_{+-} \quad . \tag{18}$$

An integration constant is defined by the condition at the horizon [11, 12]:

$$T_{--}|_{\mathrm{hor}} = 0 \tag{19}$$

We are now able to follow closely [12] where the corresponding techniques have been used for minimally coupled scalars in theories of the type (3).

Let us first study the CGHS model [8] with $a = 1$. In this particular case the residual gauge freedom of the conformal gauge can be used to set $\rho = \phi$. Then the contribution of the last two terms in (17) to T_{--} is proportional to that of the the first term, which, in turn, is the well known one for minimal coupling. Integrating these three terms in (18) with (19) yields

$$T^{(1)}_{--}|_{asymp} = \frac{\lambda^2}{48\pi}(1 - 2\alpha - \beta) \tag{20}$$

where $4\lambda^2 = B$. To evaluate the contribution of the second term in (17) we need a slightly different coordinate system [12]:

$$ds^2 = L(U)(-d\tau^2 + dz^2), \quad dU = L(U)dz \tag{21}$$

For the CGHS model [12] we have also

$$L(U) = \frac{e^{\sqrt{B}U}C}{2B} - 1\ , \qquad \phi = \sqrt{\frac{B}{4}}U \tag{22}$$

where C is a real parameter proportional to the mass of the black hole. The Killing norm L(U) vanishes at the horizon and the asymptotically flat region corresponds to $U \to -\infty$. Eq. (18) is now also easily integrated with (19), and we arrive at the the Hawking flux in the CGHS model:

$$T_{--}^{CGHS}|_{asymp} = \frac{\lambda^2}{48\pi}(1 + \frac{3}{2}\alpha^2 - 2\alpha - \beta) \tag{23}$$

Even for minimal coupling ($\alpha = 0$) this expression is inherently ambiguous due to the constant β which had its roots in the ambiguous definition of an ultralocal measure. Increasing α (non-minimal coupling) above $\alpha = 4/3$ tends to increase T_{--}. Of course, by adjusting β the flux may become zero or even negative as well ('cold dilaton black hole'). Like the (geometric) Hawking temperature the expression (23) for the CGHS model does not depend on the mass of the black hole.

For other values of a a simplification by using a residual gauge freedom is not possible. Nevertheless, starting from (21) with [7,12]

$$\begin{aligned} L(U) &= \frac{aC}{2B}\left[\frac{U^2(a-1)^2B}{a}\right]^{\frac{a}{2(a-1)}} - 1 \\ U &= \sqrt{\frac{a}{B}}\frac{e^{-2(1-a)\phi}}{a-1} \end{aligned} \tag{24}$$

the integration of (18) is rather elementary, though a bit tedious. The final result reads

$$T_{--}^{(a)}|_{asymp} = \frac{1}{48\pi}T_H^2\left(1 - \frac{3\alpha^2}{2(2-a)} - \frac{1}{2-a}(2\alpha + \beta)\right) \tag{25}$$

which has been expressed in terms of the general geometric Hawking temperature for the models (3)

$$T_H^2 = \frac{a^2}{8}C^{\frac{2(a-1)}{a}}\left(\frac{2B}{a}\right)^{\frac{2-a}{a}} \quad . \tag{26}$$

Taking the limit $a \to 1$ in (25) does not reproduce (20) since there is no smooth limit in the solutions (24). The difference in the sign of the second term in the brackets is due to a finite contribution of $T_{+-}|_{hor}$ for the CGHS model whereas this term vanishes for all other cases. For SSG all parameters are unambiguously defined ($a = \frac{1}{2}, \alpha = \beta = 1$). Then the bracket in (25) yields a factor -2, i.e. a negative flux!

4 Discussion

The trace anomaly for $T_{\mu\nu}$ in the case of non-minimally coupled scalar fields with trivial measure ($\psi = 0$) recently has been the subject of several studies [2–5]. The only difference to our result (16) is a coefficient in front of $\Box\varphi$. It should be noted that at this point these authors also disagree among themselves. The reason is that their methods [2, 3, 6] cannot fix total derivatives unambiguously. Nojiri and Odintsov define the trace anomaly through the equation $\Gamma_{div} \propto \int d^2x\sqrt{g}T^{\mu}_{\mu}$ with the correct expression [4] for the divergent part of the effective action Γ_{div}. Of course, an arbitrary total derivative can be added to T^{μ}_{μ} without changing Γ_{div}. It is interesting to note that complete agreement with our expression (16) is achieved, if in [3, 6] one replaces the expressions for scalar curvature and metric by conformally transformed ones (see (15)). Bousso and Hawking use global scale transformation to relate the coefficients in front of R and $\Box\varphi$. This method allows for some ambiguity as the authors admit themselves [2]. Mikovic and Radovanovic calculate the stress energy tensor by varying the finite part of the effective action. However, they perform scale transformations of quantum fields (see eqs. (2.4) and (2.25) in [5]). In the presence of the conformal anomaly, such transformations, in general, change the effective action. Thus some contributions can be overlooked in such an approach. In [13] it was suggested that the coefficients in front of $\Box\varphi$ are due to boundary terms of the classical action.

The crucial difference of our method is the use of a *local* scale transformation inside the zeta function. Due to the presence of an arbitrary *function* δk one cannot integrate by parts in (12) in order to remove a total derivative in (9). Hence, all terms there are fixed unambiguously.

Another attempt to calculate Hawking radiation for non-minimally coupled scalar fields was made in [6]. The physical content of the model considered there is totally different from ours and therefore will not be discussed here.

Our result for the 'anomaly', eq.(17), is the most general one obtainable in 1+1 dimensional theories. The result for the Hawking flux in SSG, on the other hand, taken literally would mean that an influx of matter is necessary to maintain in a kind of thermodynamical equilibrium the Hawking temperature of a black hole – in complete contradiction to established black hole wisdom. However, to put this result on a sound basis the treatment of Hawking radiation in the asymptotic region in that case certainly requires to go beyond the usual approach adopted also in our present paper. After all,

non-minimally coupled scalar fields are strongly coupled in the asymptotic region. Therefore a result like (25) for SSG cannot be the final answer. In fact, probably new methods for extracting the flux towards infinity in such a case have to be invented.

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References

[1] P. Thomi, B. Isaak and P. Hajicek, Phys. Rev. D **30**, 1168 (1984); P. Hajicek, Phys. Rev. D **30**, 1168 (1984).

[2] R. Bousso and S. W. Hawking, *Trace anomaly of dilaton coupled scalars in two dimensions*, `hep-th/9705236`.

[3] S. Nojiri and S. D. Odintsov, *Trace anomaly induced effective action for 2D and 4D dilaton coupled scalars*, `hep-th/9706143`.

[4] E. Elizalde, S. Naftulin and S. D. Odintsov, Phys. Rev. D **49**, 2852 (1994).

[5] A. Mikovic and V. Radovanovic, *One-loop effective action for spherical scalar field collapse*, `hep-th/9706066`.

[6] S. Nojiri and S. D. Odintsov, *Trace anomaly and nonlocal effective action for 2D conformally invariant scalar interacting with dilaton*, `hep-th/9706009`.

[7] M.O. Katanaev, W. Kummer and H. Liebl, Nucl. Phys. B **486**, 353 (1997).

[8] G. Mandal, A. Sengupta and S. R. Wadia, Mod. Phys. Lett. A **6** 1685 (1991); S. Elitzer, A. Forge and E. Rabinovici, Nucl. Phys. B **359**, 581

(1991); E. Witten, Phys. Rev. D **44**, 314 (1991); C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger, Phys. Rev. D **45**, 1005 (1992).

[9] G. Esposito, A.Y. Kamenshchik and G. Pollifrone, *Euclidean Quantum Gravity on Manifolds with Boundary*, Kluwer, 1997.

[10] P. B. Gilkey, J. Diff. Geom. **10**, 601 (1975); *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, CRC Press, Boca Raton, 1994.

[11] S. M. Christensen and S. A. Fulling, Phys. Rev. D **15**, 2088 (1977).

[12] H. Liebl, D. V. Vassilevich and S. Alexandrov, Class. Quantum Grav. **14**, 889 (1997).

[13] S. Ichinose, *Weyl Anomaly of 2D Dilaton-Scalar Gravity and Hermiticity of System Operator*, `hep-th/9707025`.